Einstein EFE

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = \frac{8\pi G}{c^4} T_{\mu\nu}$$

0.1 The metric tensor $g_{\mu\nu}$

$$d\phi = \sum_n \frac{\partial \phi}{\partial x^n} dx^n \tag{0.1}$$

Difference in frame height as sum of partial derivatives of the coordinates

$$\frac{\partial \phi}{\partial y^n} = \sum_m \frac{\partial \phi}{\partial x^m} \frac{dx^m}{dy^n} \tag{0.2}$$

Gradient of n coordinate in y frame as the sum of all the partial derivatives in x frame

0.1.1 vectorial notation

$$V_y^n = \sum_m \frac{\partial y^n}{\partial x^m} V_x^m \tag{0.3}$$

Vector transformation from frame y to frame x as the sum of the product of gradient and vector in frame x

0.1.2 tensorial notation

$$\begin{aligned} T_y^{mn} = \sum_{rs} \frac{\partial y^m}{\partial x^r} \frac{\partial y^n}{\partial x^s} A_x^r B_x^s = \\ = \sum_{rs} \frac{\partial y^m}{\partial x^r} \frac{\partial y^n}{\partial x^s} T_x^{rs} \end{aligned} \tag{0.4}$$

Contravariant tensor transformation from frame y to frame x as the sum of the product of gradients and tensor in frame x

$$T_{mn(y)} = \sum_{rs} \frac{\partial x^r}{\partial y^m} \frac{\partial x^s}{\partial y^n} T_{rs}(x) \tag{0.5}$$

Covariant tensor transformation from frame y to frame x as the sum of the product of gradients and tensor in frame x

$$ds^2 = \delta_{mn} \sum_{mn} dx^n dx^m \tag{0.6}$$

Substituting dx^m,dx^n by equation 0.1

$$
\begin{aligned}
ds^2 &= \delta_{mn} \sum_{rs} \frac{\partial x^m}{\partial y^r} dy^r \frac{\partial x^n}{\partial y^s} dy^s \\
&= \delta_{mn} \sum_{rs} \frac{\partial x^m}{\partial y^r} \frac{\partial x^n}{\partial y^s} dy^s dy^r
\end{aligned} \tag{0.7}
$$

where δ_{mn} is the Chronecker Delta
And the metric tensor is :

$$g_{mn} = \delta_{mn} \sum_{rs} \frac{\partial x^m}{\partial y^r} \frac{\partial x^n}{\partial y^s}$$

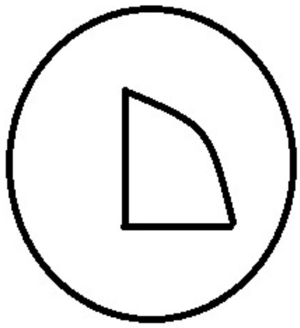

Figure 0.1: Triangle inscribed in a curved surface

The metric tensor corrects Pythagoras for curved space.
In a flat space it reduces to Cronecker Delta.
And so,

$$ds^2 = g_{mn} dy^r dy^s$$

If
$T_{mn}(x)$ is a derivative with respect to x,
$T_{mn}(x) = \frac{\partial V_m}{\partial x^n}(x)$ is it true that
$T_{mn}(y) = \frac{\partial V_m}{\partial y^n}(y)$
if $T_{mn}(y)$is the same as $T_{mn}(x)$but in different frames
$T_{mn}(x) = T_{mn}(y)$

and the answer is NO.
From eq.0.5 we substiture $T_{rs}(x)$ with $T_{mn}(x)$derivatives as above

$$T_{mn(y)} = \frac{\partial x^r}{\partial y^m}\frac{\partial x^s}{\partial y^n}T_{rs}(x) = \frac{\partial x^r}{\partial y^m}\frac{\partial x^s}{\partial y^n}\frac{\partial V_r(x)}{\partial x^s}$$

By contracting the last expression we get

$$T_{mn(y)} = \frac{\partial x^r}{\partial y^m}\frac{\partial V_r(x)}{\partial y^n}$$

and the question is

$$T_{mn(y)} = \frac{\partial x^r}{\partial y^m}\frac{\partial V_r(x)}{\partial y^n} =? \frac{\partial V_m(y)}{\partial y^n}$$

From eq.0.5 by dropping summation and for a single index we have

$$\frac{\partial V_m(y)}{\partial y^n} = \frac{\partial}{\partial y^n}(\frac{\partial x^r}{\partial y^m}V_r(x))$$

Differenciating the product by the rule

$$dAB = AdB + BdA$$

we have

$$\frac{\partial V_m(y)}{\partial y^n} = \frac{\partial x^r}{\partial y^m}\frac{\partial V_r(x)}{\partial y^n} + \frac{\partial}{\partial y^n}\frac{\partial x^r}{\partial y^m}V_r(x)$$

which is

$$T_{mn(y)} + \frac{\partial}{\partial y^n}\frac{\partial x^r}{\partial y^m}V_r(x)$$

From the additional term

$$\frac{\partial}{\partial y^n}\frac{\partial x^r}{\partial y^m}$$

is the Christoffel symbol.

$$\Gamma^r_{nm} = \frac{\partial}{\partial y^n}\frac{\partial x^r}{\partial y^m}$$

So the answer in our previously expressed question is the equality

$$T_{mn(y)} = \nabla_n V_m = \frac{\partial V_m(y)}{\partial y^n} + \Gamma^r_{nm} V_r(x) \tag{0.8}$$

which is the covariant derivative of the vector V(x). Now in terms of tensors

$$\nabla_p T_{mn} = \frac{\partial T_{mn}}{\partial y^r} + \Gamma^r_{pm} T_{rn} + \Gamma^r_{pn} T_{rm} \tag{0.9}$$

The Einstein version of Pythagoras
$ds^2 = dx^{1^2} + dx^{2^2} = \sum dx^n dx^n = \sum dx^m dx^n \delta_{mn}$

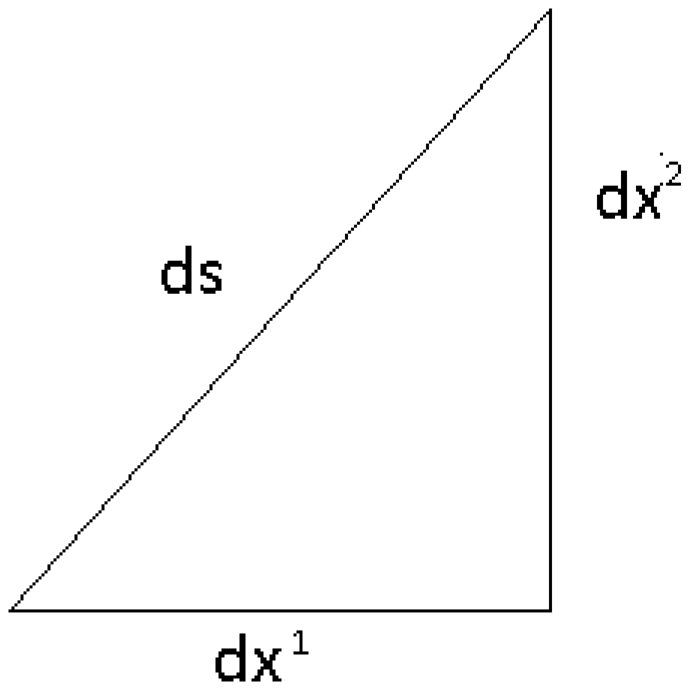

Figure 0.2: a cone

In flat space $\nabla_r g_{mn}(x) = 0$

$$\nabla_p g_{mn} = \frac{\partial g_{mn}}{\partial y^r} + \Gamma^r_{pm} g_{rn} + \Gamma^r_{pn} g_{rm} = 0 \tag{0.10}$$

$$\Gamma^a_{bc}(x) = \frac{1}{2} g^{ad} \sum \frac{\partial g_{dc}}{\partial x_b} + \frac{\partial g_{ab}}{\partial x^c} - \frac{\partial g_{bc}}{\partial x^d} \quad (0.11)$$

The Christoffel symbol in terms of a metric tensor

Figure 0.3: a cone

When you open up the cone on a flat surface,you can draw an extension of OB. Parallel transfer the extension around. At the end point A the parallel will not be parallel to B. There will be an angle difference between AOB and alpha, and that represents the curvature.

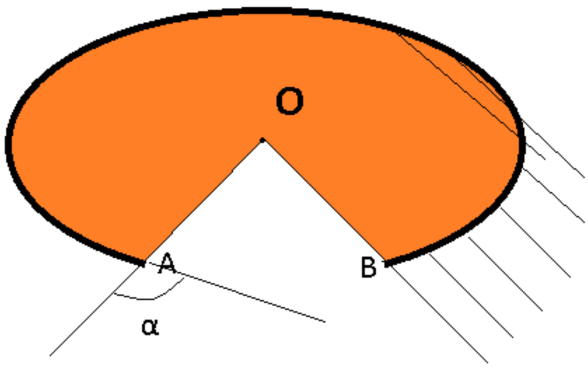

Figure 0.4: a flat cone

0.1.3 cumutators

$[A, B] = AB - BA$
Now between a derivative and a function we are going to show that
$[\frac{\partial}{\partial x}, f(x)] = \frac{\partial f(x)}{\partial x}$
$\frac{\partial}{\partial x} f(x)V - f(x)\frac{\partial}{\partial x}V$
By the differentiation of a product rule
dAB=BdA+AdB we get
$V\frac{\partial f(x)}{\partial x} + f(x)\frac{\partial V}{\partial x} - f(x)\frac{\partial V}{\partial x} = V\frac{\partial f(x)}{\partial x}$

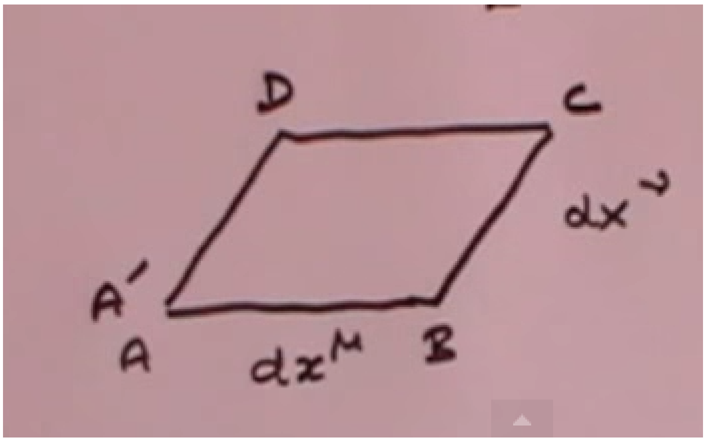

Figure 0.5: Parallel transport by vectors

If we take the vector differences in the 2 directions d_x^μ,d_x^ν
$(V_C - V_D) - (V_B - V_A)$
$(V_C - V_B) - (V_D - V_{A'})$
Now subtracting the above we get
$(V_C - V_D) - (V_B - V_A) - (V_C - V_B) - (V_D - V_{A'}) = V_A - V_{A'} = dV$
where dV is the difference in the vector V transfered in parallel. In flat space the difference will be zero because there will be no difference in vector V.
So the difference in the direction x will be:
$V_C - V_D = \frac{\partial V}{\partial x^m} dx^m V = \nabla_m dx^m V$
Now the difference of the differences in one direction will be the covariant in one direction of the covariant in the other direction
$(V_C - V_D) - (V_B - V_A) = \nabla_\nu dx^\nu \nabla_\mu dx^\mu V$
and in the other direction
$(V_C - V_B) - (V_D - V_{A'}) = \nabla_\mu dx^\mu \nabla_\nu dx^\nu V$

and now
$dv = \nabla_\nu dx^\nu \nabla_\mu dx^\mu V - \nabla_\mu dx^\mu \nabla_\nu dx^\nu =$
$dx^\mu dx^\nu V(\nabla_\nu \nabla_\mu - \nabla_\mu \nabla_\nu) =$
$= dx^\mu dx^\nu V[\nabla_\nu, \nabla_\mu]$
where $[\nabla_\nu, \nabla_\mu] = R_{\mu\nu}$=Ricci tensor
The covariant is
$\nabla_\nu = \partial_\nu + \Gamma_\nu$
where ∂_ν is $\frac{\partial}{\partial x^\nu}$
The commutator of the 2 covariants expands to
$(\partial_\nu + \Gamma_\nu)(\partial_\mu + \Gamma_\mu) - (\partial_\mu + \Gamma_\mu)(\partial_\nu + \Gamma_\nu) =$
$= (d_\nu d_\mu + \Gamma_\nu d_\mu + d_\nu \Gamma_m + \Gamma_\nu \Gamma_\mu) - (\partial_\mu \partial_\nu + \partial_\mu \Gamma_\nu + \Gamma_\mu d_\nu + \Gamma_\mu \Gamma_\nu) =$
$= 0 - [d_\mu, \Gamma_\nu] + [d_\nu, \Gamma_\mu] + [\Gamma_\nu, \Gamma_\mu] =$
$= -\frac{\partial \Gamma_\nu}{\partial x^\mu} + \frac{\partial \Gamma_\mu}{\partial x^\nu} + [\Gamma_\nu, \Gamma_\mu]$

Appendix A

Multichain rule proof in the context of equation 0.2

Since $z = f(x, y)$ is differentiable at the point (x, y),

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$. Thus,

$$\frac{\Delta z}{\Delta t} = \frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}.$$

Taking the limit as $\Delta t \to 0$,

$$\lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \lim_{\Delta t \to 0} \left[\frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}\right]$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} + \left(\lim_{\Delta t \to 0} \varepsilon_1\right)\frac{dx}{dt} + \left(\lim_{\Delta t \to 0} \varepsilon_2\right)\frac{dy}{dt}.$$

But $\lim_{\Delta t \to 0} \Delta x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}\Delta t = \frac{dx}{dt} \lim_{\Delta t \to 0} \Delta t = 0$ and similarly $\lim_{\Delta t \to 0} \Delta y = 0$, so $\lim_{\Delta t \to 0} \varepsilon_1 = \lim_{\Delta t \to 0} \varepsilon_2 = 0$.
Thus,

$$\frac{dx}{dt} = \frac{\partial x}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} + (0)\frac{dx}{dt} + (0)\frac{dy}{dt}$$

$$= \frac{\partial x}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Figure 1.1: The multivariable chain rule proof

The analogy in context of equation 0.2:

$x = x^1 = x^1(y^1)$

$y = x^2 = x^2(y^1)$

$z = f(x, y) \equiv \phi = f(x^1, x^2)$

$$x = x(t) \equiv x^1 = x^1(y^1)$$
$$y = y(t) \equiv x^2 = x^2(y^1)$$
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$
$$\equiv \frac{\partial \phi}{\partial y^1} = \frac{\partial \phi}{\partial x^1}\frac{\partial x^1}{\partial y^1} + \frac{\partial \phi}{\partial x^2}\frac{\partial x^2}{\partial y^2}$$