"Matter tells space how to curve.

Space tells matter how to move."

John Archibald Wheeler
4.1: Introduction to General Relativity

目的

- Introduce the Tools of General Relativity
- Become familiar with the Tensor environment
- Look at how we can represent curved space times
- How is the geometry represented?
- How is the matter represented?
- Derive Einstein’s famous equation
- Show how we arrive at the Friedmann Equations via General Relativity
- Examine the Geometry of the Universe

Suggested Reading on General Relativity
- Introduction to Cosmology - Narlikar 1993
- Cosmology - The origin and Evolution of Cosmic Structure - Coles & Lucchin 1995
- Principles of Modern Cosmology - Peebles 1993
Newton v Einstein

Newton:
• Mass tells Gravity how to make a Force
• Force tells mass how to accelerate

Einstein:
• Mass tells space how to curve
• Space tells mass how to move

Newton:
• Flat Euclidean Space
• Universal Frame of reference

Einstein:
• Space can be curved
• Its all relative anyway!!
4.1: Introduction to General Relativity

The Principle of Equivalence

Recall: Equivalence Principle \( m_g = m_i \)

A more general interpretation led Einstein to his theory of General Relativity

PRINCIPLE OF EQUIVALENCE:
An observer cannot distinguish between a local gravitational field and an equivalent uniform acceleration

Implications of Principle of Equivalence

- Imagine 2 light beams in 2 boxes
  - the first box being accelerated upwards
  - the second in freefall in gravitational field
- For a box under acceleration
  - Beam is bent downwards as box moves up
- For box in freefall photon path is bent down!!

Fermat’s Principle:
- Light travels the shortest distance between 2 points
- Euclid = straight line
- Under gravity - not straight line
- Space is not Euclidean!
4.2: Geometry, Metrics & Tensors

- The Interval (line element) in Euclidean Space:

\[ ds^2 = (dx^2 + dy^2)^{1/2} \]

2-D \[ ds^2 = dx^2 + dy^2 \]

3-D \[ ds^2 = dx^2 + dy^2 + dz^2 \]

2-D \[ ds^2 = dr^2 + r^2 d\theta^2 \]

3-D \[ ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \]
4.2: Geometry, Metrics & Tensors

- The Interval in Special Relativity (The Minkowski Metric):

- The Laws of Physics are the same for all inertial Observers (frames of constant velocity)
- The speed of light, c, is a constant for all inertial Observers
  - Events are characterized by 4 co-ordinates (t, x, y, z)
  - Length Contraction, Time Dilation, Mass increase
  - Space and Time are linked
    - The notion of SPACE-TIME

\[ ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2) \]

\[ ds^2 = c^2 dt^2 - \left[ dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right] \]
4.2: Geometry, Metrics & Tensors

- The Interval in Special Relativity (The Minkowski Metric):

\[ ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2) \]

- Equation of a light ray, \( ds^2 = 0 \), \( \frac{dx}{dt} = \pm c \)
  - Trace out light cone from Observer in Minkowski S-T
  - Spreading into the future
  - Collapsing from the past

- Area within light cone: \( ds^2 > 0 \)
  - Events that can affect observer in past, present, future.
  - This is a timelike interval.
  - Observer can be present at 2 events by selecting an appropriate speed.

- Area outside light cone: \( ds^2 < 0 \)
  - Events that are causally disconnected from observer.
  - This is a spacelike interval.
  - These events have no effect on observer.
4.2: Geometry, Metrics & Tensors

• The Interval in General Relativity:

  In General

  The interval is given by:

  \[ ds^2 = \sum_{i,j=0}^{n} g_{ij} dx^i dx^j \]

  • \( g_{ij} \) is the metric tensor (Riemannian Tensor):
  • Tells us how to calculate the distance between 2 points in any given spacetime
  • Components of \( g_{ij} \)
    • Multiplicative factors of differential displacements (\( dx^i \))
    • Generalized Pythagorean Theorem
4.2: Geometry, Metrics & Tensors

• The Interval in General Relativity:

\[
\text{ds}^2 = \sum_{i,j=0}^{n} g_{ij} dx^i dx^j
\]

**Euclidean Metric**

\[
ds^2 = dx^2 + dy^2 + dz^2 \quad (n = 2) \quad 3 \text{ Dimensions}
\]

\[
dx^1 = x
\]

\[
g_{i,j} = W_{i,j}
\]

\[
dx^2 = y
\]

\[
W_{i,j} = 0 \quad (i \neq j)
\]

\[
dx^3 = z
\]

\[
g_{i,j} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

**Minkowski Metric**

\[
ds^2 = c dt^2 \sum (dx^2 + dy^2 + dz^2) \quad (n = 3) \quad 4 \text{ Dimensions}
\]

\[
dx^0 = t \quad dx^1 = x
\]

\[
dx^2 = y \quad dx^3 = z
\]

\[
g_{i,j} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
4.2: Geometry, Metrics & Tensors

A little bit about Tensors - 1

Consider a matrix \( M = (m_{ij}) \)

Tensor \sim \text{arbitrary number of indices (rank)}
- Scalar = zeroth rank Tensor
- Vector = 1st rank Tensor
- Matrix = 2nd rank Tensor: matrix is a tensor of type (1,1), i.e. \( m^i_j \)

Tensors can be categorized depending on certain transformation rules:
- Covariant Tensor: indices are low
- Contravariant Tensor: indices are high
- Tensor can be mixed rank (made from covariant and contravariant indices)
- Euclidean 3D space: Covariant = Contravariant = Cartesian Tensors
- 注意 \( m^i_{jk} \neq m^i_{ik} \)
4.2: Geometry, Metrics & Tensors

- A little bit about Tensors - 2

Covariant Tensor Transformation:

Consider Scalar quantity \( \Box \), Scalar \( \Box \) invariant under coordinate transformations

\[ \Box(x_i) = \Box(x'_i) \]

\[ \Box(x_1, x_2, x_3) = \Box(x'_1, x'_2, x'_3) \]

Taking the gradient \( \nabla = \frac{\partial \Box}{\partial x_1} \hat{\mathbf{x}}_1 + \frac{\partial \Box}{\partial x_2} \hat{\mathbf{x}}_2 + \frac{\partial \Box}{\partial x_3} \hat{\mathbf{x}}_3 \)

this normal has components \( A_i = \frac{\partial \Box}{\partial x_i} \)

Which should not change under coordinate transformation, so

\[ A'_i = \frac{\partial \Box}{\partial x'_i} = \frac{\partial \Box}{\partial x'_i} \]

Also have the relation

\[ \frac{\partial \Box}{\partial x'_i} = \frac{\partial \Box}{\partial x_j} \frac{\partial x_j}{\partial x'_i} \]

So for,

\[ A_i = \frac{\partial \Box}{\partial x_i} \quad A'_i = \frac{\partial \Box}{\partial x'_i} \quad A_j = \frac{\partial \Box}{\partial x_j} \quad \text{etc...} \]

\( A'_i = \frac{\partial \Box}{\partial x'_i} = \frac{\partial x_j}{\partial x'_i} \frac{\partial \Box}{\partial x_j} \quad A_j = a_{i,j} A_j \quad a_{i,j} = \frac{\partial x_j}{\partial x'_i} \)

Generalize for 2nd rank Tensors,

Covariant 2nd rank tensors are animals that transform as :-

\[ A'_{ij} = \frac{\partial x_k}{\partial x'_i} \frac{\partial x_l}{\partial x'_j} A_{kl} \]
4.2: Geometry, Metrics & Tensors

- A little bit about Tensors - 3

Contravariant Tensor Transformation:

Consider a curve in space, parameterized by some value, \( r \)

Coordinates of any point along the curve will be given by: \( x^i(r) \)

Tangent to the curve at any point is given by: \( A^i = \frac{dx^i}{dr} \)

The tangent to a curve should not change under coordinate transformation, so \( A'^i = \frac{dx'^i}{dr} \)

Also have the relation

\[
\frac{dx'^i}{dr} = \frac{\partial x'^i}{\partial x_j} \frac{dx^j}{dr}
\]

So for, \( A^i = \frac{dx^i}{dr} \) \( A'^i = \frac{dx'^i}{dr} \) \( A^j = \frac{dx^j}{dr} \) etc......

\[
A'^i = \frac{dx'^i}{dr} = \frac{\partial x'^i}{\partial x_j} \frac{dx^j}{dr} = \frac{\partial x'^i}{\partial x_j} A^j
\]

\[
A'^i = a^{i,j} A^j
\]

\[
a^{i,j} = \frac{\partial x'^i}{\partial x^j}
\]

Generalize for 2nd rank Tensors,

Contravariant 2nd rank tensors are animals that transform as :-

\[
A'^{ij} = \frac{\partial x'^i}{\partial x_k} \frac{\partial x'^j}{\partial x_l} A^{kl}
\]
4.2: Geometry, Metrics & Tensors

- More about Tensors - 4
  - Index Gymnastics

- Raising and Lowering indices:

\[ g_{ij} A^i = A_j \]

\[ g^{ij} A_i = A^j \]

- Einstein Summation:

Repeated indices (in sub and superscript) are implicitly summed over

\[ a_i a_i = a_i a_i \]

\[ A_{ik} B^k = A_{ik} B^k \]

\[ A_{ik} B^i B^k = A_{ik} B^i B^k \]

Einstein c.1916: "I have made a great discovery in mathematics; I have suppressed the summation sign every time that the summation must be made over an index which occurs twice..."

Thus, the metric \[ ds^2 = \sum_{i,j=0}^{n} g_{ij} dx^i dx^j \] May be written as \[ ds^2 = g_{ij} dx^i dx^j \]
4.2: Geometry, Metrics & Tensors

- More about Tensors - 5
  - Tensor Manipulation

- Tensor Contraction:
  Set unlike indices equal and sum according to the Einstein summation convention. Contraction reduces the tensor rank by 2.
  For a second-rank tensor this is equivalent to the Scalar Product
  \[ A^i B_j = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3 \]

Minkowski Spacetime (Special Relativity)

\[ g_{ij} A^i A^j < 0 \quad \text{spacelike} \]
\[ g_{ij} A^i A^j > 0 \quad \text{timelike} \]
\[ g_{ij} A^i A^j = 0 \quad \text{null} \]
4.2: Geometry, Metrics & Tensors

- More about Tensors - 6
  - Tensor Calculus

- The Covariant Derivative of a Tensor:

  Derivatives of a Scalar transform as a vector

  How about derivative of a Tensor?? Does \( \frac{\partial A_i}{\partial x^i} \) transform as a Tensor?

  Let's see......

  Take our definition of Covariant Tensor

  \[
  A'_i = \frac{\partial x^j}{\partial x'^i} A_j
  \]

  \[
  \frac{\partial A'_i}{\partial x'^m} = \frac{\partial x^j}{\partial x'^i} \frac{\partial x^n}{\partial x'^m} \frac{\partial A_j}{\partial x^n} + \frac{\partial^2 x^j}{\partial x'^m} \frac{\partial x^n}{\partial x'^i} A_j
  \]

  Problem: Implies transformation coefficients vary with position

  Correction Factor: \( \Box_{ij}^k \) such that for \( \frac{\partial A_i}{\partial x^j} \), \( \Box^k A_k \Box^j \)

  Redefine the covariant derivative of a tensor as:

  \[
  A_{i,j} = \frac{\partial A_i}{\partial x^j} \Box_{ij}^k A_k \equiv A_{i,j} \Box_{ij}^k A_k
  \]

  - Christoffel Symbols
  - Defines parallel vectors at neighbouring points
  - Parallelism Property: affine connection of S-T
  - Christoffel Symbols = fn(spacetime), normal derivative; covariant derivative
4.2: Geometry, Metrics & Tensors

- **Riemann Geometry**

Covariant Differentiation of Metric Tensor

\[ g_{ik;l} = g_{ik,l} \square_i^p g_{pk} \square_l^p g_{ip} \]

General Relativity formulated in the non-Euclidean Riemann Geometry

- **Simplifications**
  \[ \square_i^k = \square_i^k \]
  \[ g_{ik;l} = 0 \]

- **Equations**
  40 linear equations - ONE Unique Solution for Christoffel symbol

\[ \square_i^k = \frac{g^{im}}{2} \left( g_{mk,l} + g_{ml,k} \square g_{kl,m} \right) \]

**SPECIAL RELATIVITY**

\[ ds^2 = c^2 dt^2 \square (dx^2 + dy^2 + dz^2) \]

\[ ds = \text{constant} \] for path

**GENERAL RELATIVITY**

\[ ds^2 = g_{ij} dx^i dx^j \]

\[ ds = 0 \] for path

Gravity is a property of Spacetime, which may be curved

Path of a free particle is a geodesic where

\[ \frac{d^2 x^i}{ds^2} + \square_{kl}^i \frac{dx^k}{ds} \frac{dx^l}{ds} = 0 \]

But lines are not straight (because of the metric tensor)
4.2: Geometry, Metrics & Tensors

• The Riemann Tensor

Recall

The Riemann Tensor

\[ \square A_i = \square^l_{ik} A_i \square^k_l \]

Such that we needed to define the Covariant derivative

\[ A_{ik} = \frac{\partial A_i}{\partial x_k} \square^l_{ik} A_l \equiv A_{i,k} \square^l_{ik} A_l \]

To transport a vector without the result being dependent on the path,

Require a vector \( A_i(x^k) \) such that

\[ \frac{\partial A_i}{\partial x^k} = \square^l_{ik} A_l \]

Interchange differentiation w.r.t. \( x_n \) & \( x_k \) and use \( A_{i, nk} = A_{i, kn} \)

Necessary condition for

\[ 0 = \frac{\partial}{\partial x^n} \square^m_{ik} \frac{\partial}{\partial x^m} \square^m_{ln} \]

Can show: \( A_{i, nk} \square A_{i, kn} \equiv R^m_{i, kn} A_m \) Where RHS=LHS=Tensor \( R^m_{i, kn} \) is the Riemann Tensor

• Defines geometry of spacetime (=0 for flat spacetime).
• Has 256 components but reduces to 20 due to symmetries.
4.2: Geometry, Metrics & Tensors

• The Einstein Tensor

• Lowering the second index of the Riemann Tensor, define

\[ g_{ij} R_{klm}^j = R_{iklm} \]

• Contracting the Riemann Tensor gives the Ricci Tensor describing the curvature

\[ R_{kl} = g^{im} R_{iklm} \equiv R^m_{klm} \]

• Contracting the Ricci Tensor gives the Scalar Curvature (Ricci Scalar)

\[ R = g_{kl} R_{kl} \]

\[ R^m_{ikn} \text{ is the Riemann Tensor} \]

• Define the Einstein Tensor as

\[ G_{ik} = R_{kl} \sum k \frac{1}{2} g_{ik} R \]

• Symmetry properties of \( R_{iklm} \)

\[ R_{iklm} + R_{iknl} + R_{ikmn} + R_{ikmn} = 0 \]

Bianchi Identities

Multiplier by \( g^{im} g^{kn} \) and using above relations

\[ R^{ik} \sum 1 \frac{1}{2} g_{ik} R k \equiv 0 \]

or...... The Einstein Tensor has ZERO divergence
4.3: Einstein’s Theory of Gravity

• The Energy Momentum Tensor

Define Energy Momentum Tensor

\[ T^{ik} = \begin{pmatrix} T^{00} \\ T^{i0} \\ T^{ik} \end{pmatrix} \]

- Density of mass/unit vol.
- Density of j\textsuperscript{th} component of momentum/unit vol.
- k-component of flux of j component of momentum

\[ T^{ik} \text{ The flux of the } i \text{ momentum component across a surface of constant } x^k \]

• Conservation of Energy and momentum - \( T^k_{i;k} = 0 \)

Non relativistic particles (Dust)

- \( \Box_o = \Box_o(x) \) - Proper density of the flow
- \( u^i = dx^i/dt \) - 4 velocity of the flow

In comoving rest frame of N dust particles of number density, \( n \)
Consider Units

- Density, \( \Box = \text{mass/vol} \)
- Momentum, \( p = \text{energy/vol} \)

In general, \( T = p \) \quad \( u = mnu \) \quad \( u = \Box u \) \quad \( u \)

- In any general Lorentz frame \( T^{ik} = \Box u^i u^k \)

Dust approximation
World lines almost parallel
Speeds non-relativistic

Looks like classical Kinetic energy

\[ T^{ik} = T^{00} = \frac{\text{Vol}}{N} m_N c^2 = nmc^2 = \Box_o c^2 \]
4.3: Einstein’s Theory of Gravity

- The Energy Momentum Tensor

Relativistic particles (inc. photons, neutrinos)

Particle 4 Momentum, $p^i = \frac{E}{c}, p^i$

$E^2 = c^2 p^2 + m^2 c^4 \square p^2 c^2$

\[ T^{00} = \frac{\text{Vol}}{N} E_N = \square \quad \square - \text{energy density} \]

\[ T^{11} = T^{22} = T^{22} = \frac{\text{Vol}}{N} \frac{p^2 c^2}{3E} \square \frac{1}{3} \square \]

- Perfect Fluid

General Particle 4 Momentum,

$\vec{p}^0 = \sqrt{m c^2}$

$\vec{p}^{\square} = \sqrt{m \vec{v}} \quad (\square = 1, 2, 3)$

In rest frame of particle

\[ T^{00} = \frac{\text{Vol}}{N} \sqrt{m c^2} \square \square c^2 \quad T^{11} = T^{22} = T^{22} = \frac{\text{Vol}}{N} \frac{p^2 c^2}{3} \square P \]

For any reference frame with fluid 4 velocity = $u$

\[ T^{ik} = (P + \sqrt{c^2}) u^i u^k \square g^{ik} P \quad \text{Pressure is important} \]
4.3: Einstein’s Theory of Gravity

- The Einstein Equation

\[ T^{ik} = (P + \frac{\Box c^2}{c^2}) u^i u^k \Box g^{ik} P \]

\[ G_{ik} = R_{kl} \frac{1}{2} g_{ik} R \]

Energy Momentum Tensor (matter in the Universe)

Einstein Tensor (Geometry of the Universe)

After trial and error !!
Einstein proposed his famous equation

- Classical limit
- must reduce to Poisson’s eqn. for gravitational potential

\[ \Box = \frac{8\Box G}{c^4} \]

Basically: Relativistic Poisson Equation
4.3: Einstein's Theory of Gravity

- The Einstein Equation

"Matter tells space how to curve. Space tells matter how to move."

Fabric of the Spacetime continuum and the energy of the matter within it are interwoven

\[
G_{ik} = R_{kl} \frac{1}{2} g_{ik} R - \frac{8\pi G}{c^4} T^{ik}
\]

Does not give a static solution!!

Einstein inserted the Cosmological Constant term

"The Biggest Mistake of My Life"
4.3: Einstein’s Theory of Gravity

- **Solutions of the metric**
  \[ ds^2 = \sum_{i,j=0}^{3} g_{i,j} dx^i dx^j \] just spatial coords

- **Cosmological Principle:**
  - Isotropy \( g_{\Box,\Box} = g_{\Box,\Box} \) only take \( \Box=\Box \) terms
  - Homogeneity \( g_{0,\Box} = g_{0,\Box} = 0 \)

The Metric:
\[
\Box ds^2 = g_{00}(dx^0)^2 + g_{\Box,\Box} dx^\Box dx^\Box = g_{00}(dx^0)^2 + dl^2
\]

Proper time interval:
\[
dt = \sqrt{\frac{g_{00}}{c^2}} \Box ds^2 = c^2 dt^2 \Box dl^2
\]

How about \( g_{\Box,\Box} dx^\Box dx^\Box \)?

embed our curved, isotropic 3D space in hypothetical 4-space.

our 3-space is a hypersurface defined by \( (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = a^2 \)

length element becomes:
\[
dl^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2
\]

Constrain \( dl \) to lie in 3-Space (eliminate \( x^4 \))
\[
dl^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + \frac{x^1 dx^1 + x^2 dx^2 + x^3 dx^3}{a^2 \Box (x^1)^2 \Box (x^2)^2 \Box (x^3)^2}
\]
4.3: Einstein's Theory of Gravity

- **Solutions of the metric**

\[
\begin{align*}
x^1 &= r \sin \theta \cos \phi \\
x^2 &= r \sin \theta \sin \phi \\
x^3 &= r \cos \theta
\end{align*}
\]

Introduce spherical polar identities in 3 Space

\[
dl^2 = dr^2 + r^2 \sin^2 \theta d\phi^2 + r^2 d\theta^2 + \frac{r^2 dr^2}{a^2 r^2} = \frac{dr^2}{1 - \frac{kr^2}{a^2}} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)
\]

I. Multiply spatial part by arbitrary function of time \( R(t) \): wont affect isotropy and homogeneity because only a \( f(t) \)

II. Absorb elemental length into \( R(t) \cdot r \) becomes dimensionless

III. Re-write \( a^{-2} = k \)

**dS**

\[
dS^2 = c^2 dt^2 R^2(t) \frac{dr^2}{kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)
\]

The Robertson-Walker Metric

- \( r, \theta, \phi \) are co-moving coordinates and don’t changed with time - They are SCALED by \( R(t) \)
- \( t \) is the cosmological proper time or cosmic time - measured by observer who sees universe expanding around him
- The co-ordinate, \( r \), can be set such that \( k = -1, 0, +1 \)
4.3: Einstein’s Theory of Gravity

- The Robertson-Walker Metric and the Geometry of the Universe

\[ dS^2 = c^2 dt^2 - R^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right) \]

The Robertson-Walker Metric

\( k \sim \) defines the curvature of space time

- \( k = 0 \) : Flat Space

- \( k = -1 \) : Hyperbolic Space

- \( k = +1 \) : Spherical Space
4.3: Einstein’s Theory of Gravity

- The Robertson-Walker Metric and the Geometry of the Universe

\[ k = 0 \quad : \text{Flat Space} \]
\[ k = -1 \quad : \text{Hyperbolic Space} \]
\[ k = +1 \quad : \text{Spherical Space} \]
4.3: **Einstein's Theory of Gravity**

- **The Friedmann Equations in General Relativistic Cosmology**

Solve Einstein's equation:

$$ G_{ik} = R_{kl} - \frac{1}{2} g_{ik} R = \frac{8\pi G}{c^4} T^{ik} $$

Need:
- metric $g_{ik}$
- energy tensor $T^{ik}$

Metric is given by R-W metric:

$$ dS^2 = c^2 dt^2 - S^2(t) \left( \frac{dr^2}{kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right) $$

Co-ordinates

$$ x^0 = ct \quad x^1 = r \quad x^2 \quad x^3 $$

Non-zero $g^{ik}$

$$ g_{00} = 1 \quad g_{11} = \frac{S^2}{kr^2} \quad g_{22} = S^2 r^2 \quad g_{33} = S^2 r^2 \sin^2 \theta $$

$$ g_{ik} = \frac{1}{g^{ik}} \quad g_{00} = 1 \quad \sqrt{g} = \frac{Sr \sin \theta}{\sqrt{kr^2}} $$

Non-zero $\Gamma_{kl}^{i}$

$$ \Gamma_{01}^{0} = \frac{S}{c} \quad \Gamma_{02}^{0} = \frac{S r^2}{c} \quad \Gamma_{03}^{0} = \frac{S r^2 \sin^2 \theta}{c} $$

$$ \Gamma_{11}^{0} = \frac{kr}{kr^2} \quad \Gamma_{12}^{0} = \frac{r(1 - kr^2)}{c} \quad \Gamma_{13}^{0} = \frac{r(1 - kr^2) \sin^2 \theta}{c} $$

$$ \Gamma_{22}^{0} = \frac{1}{r} \quad \Gamma_{22}^{3} = \sin \theta \cos \theta \quad \Gamma_{23}^{3} = \cot \theta $$
4.3: Einstein’s Theory of Gravity

- The Friedmann Equations in General Relativistic Cosmology

Non-zero components of Ricci Tensor $R_{ik}$

\[
R_0^0 = \frac{3}{c^2} \frac{\ddot{S}}{S}, \quad R_1^1 = R_2^2 = R_3^3 = \frac{1}{c^2} \frac{\dddot{S}}{S} + \frac{2}{S^2} \left( \frac{\dot{S}^2 + 2kc^2}{S} \right)
\]

The Ricci Scalar $R$

\[
R = \frac{6}{c^2} \frac{\dddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2}
\]

\[
G_0^0 = R_0^0 \frac{R}{2} = \frac{3}{c^2} \frac{\dddot{S}^2 + kc^2}{S^2}
\]

Non-zero components of Einstein Tensor $G_{ik}^k$

\[
G_1^1 = G_2^2 = G_3^3 \equiv R_1^1 \frac{R}{2} = \frac{1}{c^2} \frac{\dddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2}
\]

\[
\frac{\dot{S}^2 + kc^2}{S^2} = \frac{8}{3c^2} \frac{G}{T_0^0}
\]

\[
2 \frac{S}{S} + \frac{\dot{S}^2 + kc^2}{S^2} = \frac{8}{3c^2} \frac{G}{T_1^1} = \frac{8}{3c^2} \frac{G}{T_2^2} = \frac{8}{3c^2} \frac{G}{T_3^3}
\]
4.3: Einstein’s Theory of Gravity

- The Friedmann Equations in General Relativistic Cosmology

Non-zero components of Energy Tensor \( T_{ik} \)

1. \[
2 \frac{\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2} = \frac{8\Box G}{3c^2} T_1^1 = \frac{8\Box G}{3c^2} T_2^2 = \frac{8\Box G}{3c^2} T_3^3
\]

2. \[
\frac{\dot{S}^2 + kc^2}{S^2} = \frac{8\Box G}{3c^2} T_0^0
\]

\( = p \)

\[
\frac{\partial}{\partial t} \frac{d([S^3])}{dS} + 3PS^2 = 0
\]

Assume Dust:

- \( P = 0 \)
- \( \Box = \Box c^2 \)

\[
\frac{d([S^3])}{dS} = 0 \quad \Box = \Box_o \frac{S_o}{S} \quad T_0^0 = \Box_o c^2 \frac{S_o}{S} \quad T_1^1 = 0
\]

3. \[
2 \frac{\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2} = 0
\]

\[
\frac{\dot{S}^2 + kc^2}{S^2} = \frac{8\Box G}{3c^2} \frac{S_o}{S}
\]
4.3: Einstein's Theory of Gravity

The Friedmann Equations in General Relativistic Cosmology

1. \[ \frac{\dddot{S}}{S} + \frac{\ddot{S}^2 + kc^2}{S^2} = 0 \]

2. \[ \frac{\dot{S}^2 + kc^2}{S^2} = \frac{8\tilde{G}o}{3c^2} \frac{S_o^3}{S} \]

3. \[ \square = \frac{R_o^3}{R} \]

Finally ..........

Our old friends the Friedmann Equations
4.4: Summary

- The Friedmann Equations in General Relativistic Cosmology

- We have come along way today!!!

Deriving the necessary components of The Einstein Field Equation
- Spacetime and the Energy within it are symbiotic
- The Einstein equation describes this relationship

\[ G_{ik} = R_{kl} \frac{1}{2} g_{ik} R = \frac{8 \Box G}{c^4} T^{ik} \]

\[ dS^2 = c^2 dt^2 - R^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right) \]

The Robertson-Walker Metric defines the geometry of the Universe

\[ R = \frac{4 \Box G}{3} - \frac{R}{3} \]

\[ R^2 = \frac{8 \Box G}{3} R^2 - \frac{k c^2}{3} + \frac{\Box R^2}{3} \]

The Friedmann Equations describe the evolution of the Universe

THESE WILL BE OUR TOOLBOX FOR OUR COSMOLOGICAL STUDIES
4.4: Summary

Fundamental Cosmology
4. General Relativistic Cosmology

次

Fundamental Cosmology
5. The Equation of state & the Evolution of the Universe