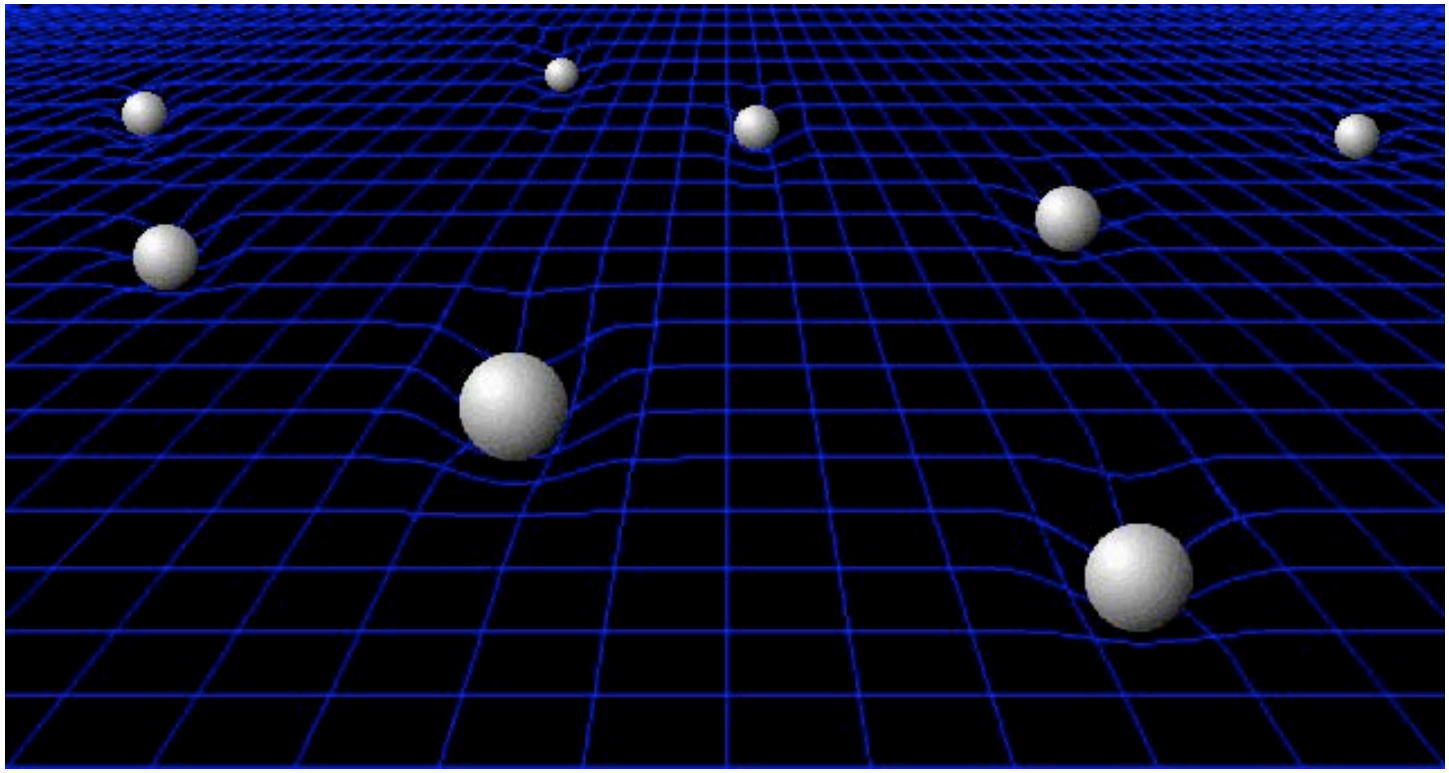
GENERAL RELATIVITY

Fundamental Cosmology: 4. General Relativistic Cosmology



"Matter tells space how to curve.

Space tells matter how to move."

John Archibald Wheeler

4.1: Introduction to General Relativity

目的

- Introduce the Tools of General Relativity
- Become familiar with the Tensor environment
- Look at how we can represent curved space times
- How is the geometry represented ?
- How is the matter represented ?
- Derive Einstein's famous equation
- Show how we arrive at the Friedmann Equations via General Relativity
- Examine the Geometry of the Universe
- このセミナーの後で皆さんは私のことが大嫌いになるかな～～～～

Suggested Reading on General Relativity

- Introduction to Cosmology - Narlikar 1993
- Cosmology - The origin and Evolution of Cosmic StructureColes & Lucchin 1995
- Principles of Modern Cosmology - Peebles 1993

4.1: Introduction to General Relativity

Newton v Einstein

Newton:
- Mass tells Gravity how to make a Force
- Force tells mass how to accelerate

Einstein:
- Mass tells space how to curve
- Space tells mass how to move

Newton:
- Flat Euclidean Space
- Universal Frame of reference

Einstein:
- Space can be curved
- Its all relative anyway!!

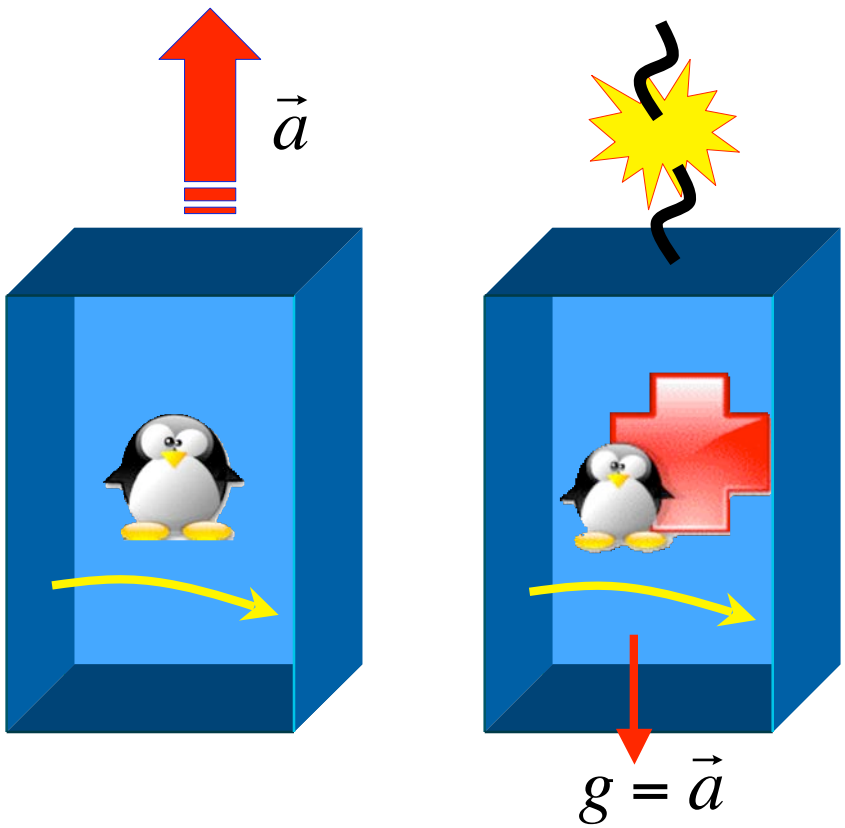
4.1: Introduction to General Relativity

The Principle of Equivalence

Recall: Equivalence Principle $m_g = m_i$

A more general interpretation led Einstein to his theory of General Relativity

PRINCIPLE OF EQUIVALENCE:
An observer cannot distinguish between a local gravitational field and an equivalent uniform acceleration



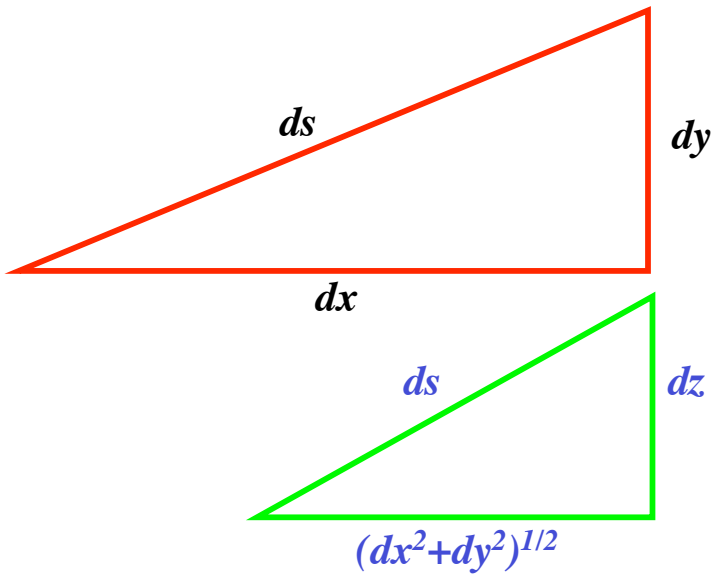
Implications of Principle of Equivalence

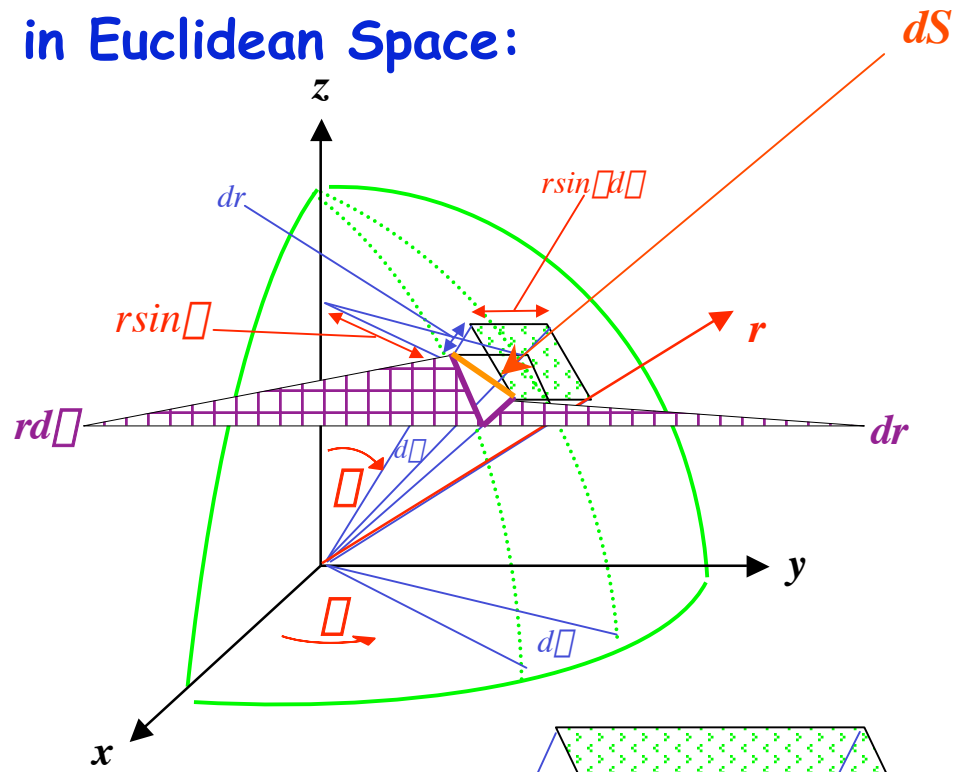
- Imagine 2 light beams in 2 boxes
  - the first box being accelerated upwards
  - the second in freefall in gravitational field
- For a box under acceleration
  - Beam is bent downwards as box moves up
- For box in freefall ➜ photon path is bent down!!

Fermat's Principle:
- Light travels the shortest distance between 2 points
- Euclid = straight line
- Under gravity - not straight line
- ➜ Space is not Euclidean !

4.2: Geometry, Metrics & Tensors

- **The Interval (line element) in Euclidean Space:**



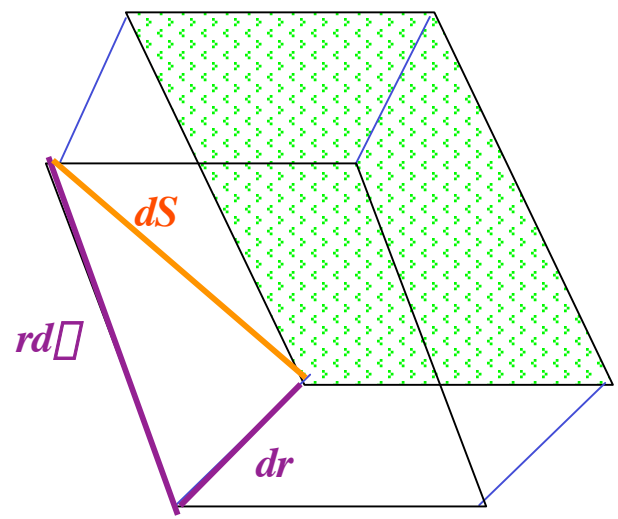


2-D $ds^2 = dx^2 + dy^2$

3-D $ds^2 = dx^2 + dy^2 + dz^2$

2-D $ds^2 = dr^2 + r^2 d\square^2$

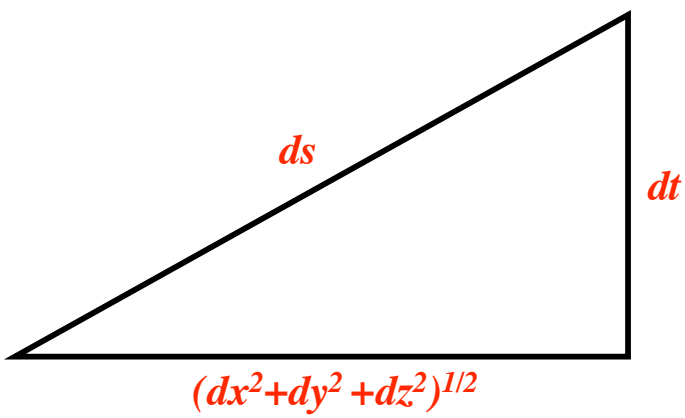
3-D $ds^2 = dr^2 + r^2 d\square^2 + r^2 Sin^2\square d\square^2$



4.2: Geometry, Metrics & Tensors

- **The Interval in Special Relativity (The Minkowski Metric) :**

- The Laws of Physics are the same for all inertial Observers (frames of constant velocity)
- The speed of light, *c*, is a constant for all inertial Observers
  - ➠ Events are characterized by 4 co-ordinates (t,x,y,z)
  - ➠ Length Contraction, Time Dilation, Mass increase
  - ➠ Space and Time are linked
    - ✰ The notion of ***SPACE-TIME***



$$ds^2 = c^2dt^2 - (dx^2 + dy^2 + dz^2)$$

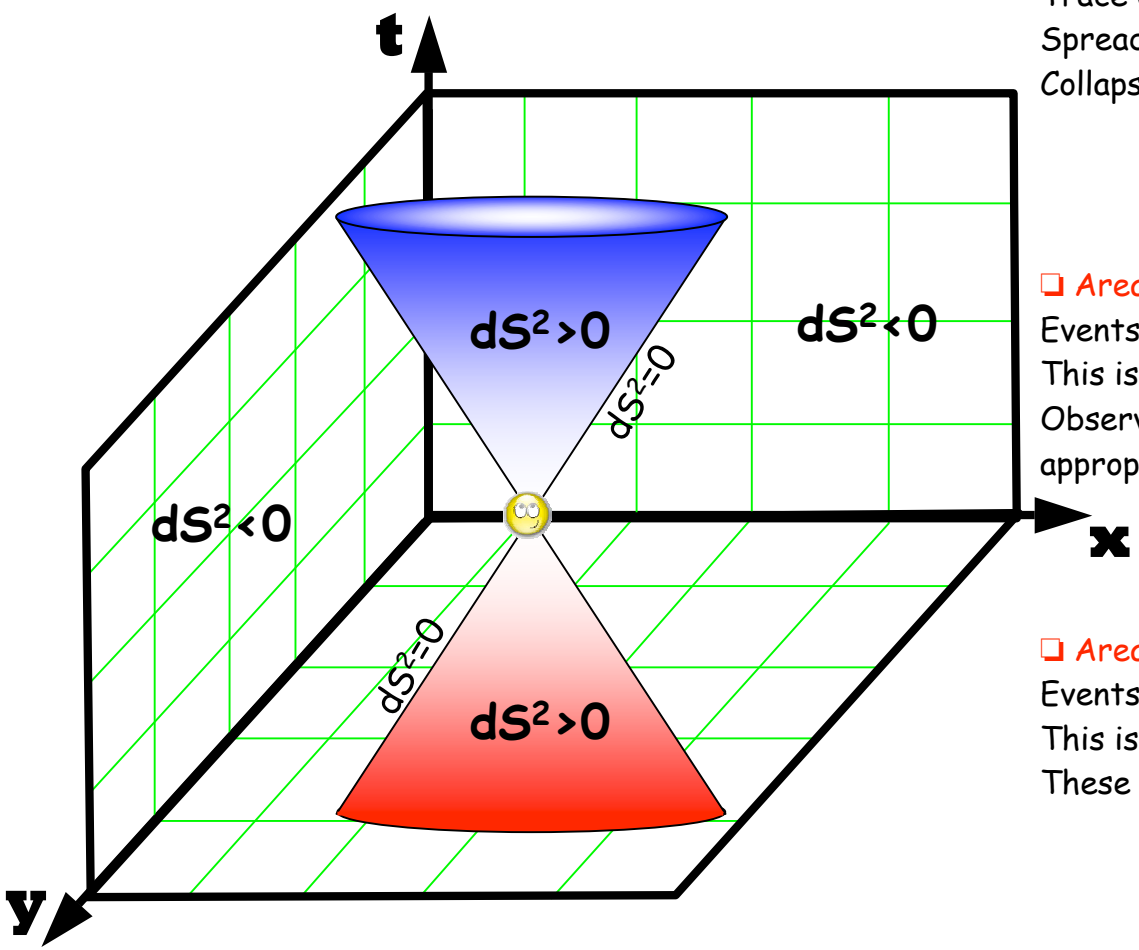
$$ds^2 = c^2dt^2 - \left[dr^2 + r^2d\theta^2 + r^2Sin^2\theta d\phi^2\right]$$

4.2: Geometry, Metrics & Tensors

- **The Interval in Special Relativity (The Minkowski Metric) :**

$$ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$$

Causally Connected Events in Minkowski Spacetime



❑ Equation of a light ray, $dS^2=0$, $\frac{dx}{dt} = \pm c$

Trace out light cone from Observer in Minkowski S-T
Spreading into the future
Collapsing from the past

❑ Area within light cone: $dS^2>0$

Events that can affect observer in past,present,future.
This is a timelike interval.
Observer can be present at 2 events by selecting an appropriate speed.

❑ Area outside light cone: $dS^2<0$

Events that are causally disconnected from observer.
This is a spacelike interval.
These events have no effect on observer.

4.2: Geometry, Metrics & Tensors

- **The Interval in General Relativity :**

❑ In General

The interval is given by:

$$\boxed{ds^2 = \sum_{i,j=0}^{n} g_{ij} dx^i dx^j}$$

- g_{ij} is the *metric tensor* (Riemannian Tensor) :
- Tells us how to calculate the distance between 2 points in any given spacetime
- Components of g_{ij}
  - Multiplicative factors of differential displacements (dx^i)
  - Generalized Pythagorean Theorem

4.2: Geometry, Metrics & Tensors

- **The Interval in General Relativity :**

$$\boxed{ds^2 = \sum_{i,j=0}^{n} g_{ij}\, dx^i dx^j}$$

Euclidean Metric $\quad ds^2 = dx^2 + dy^2 + dz^2 \qquad (n = 2)$ 3 Dimensions

$$dx^1 = x$$

$$dx^2 = y$$

$$dx^3 = z$$

$$g_{i,j} = \delta_{i,j}$$

$$\delta_{i,j} = 0 \quad (i \neq j)$$

$$g_{i,j} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Minkowski Metric $\quad ds^2 = cdt^2 - (dx^2 + dy^2 + dz^2) \qquad (n = 3)$ 4 Dimensions

$$dx^0 = t \quad dx^1 = x$$

$$dx^2 = y \quad dx^3 = z$$

$$g_{i,j} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4.2: Geometry, Metrics & Tensors

- **A little bit about Tensors - 1**

$$ds^2 = \sum_{i,j=0}^{n} g_{ij} dx^i dx^j$$

❑ Consider a matrix $M=(m_{ij})$

❑ Tensor ~ arbitray number of indicies (rank)
- Scalar = zeroth rank Tensor
- Vector = 1st rank Tensor
- Matrix = 2nd rank Tensor : matrix is a tensor of type (1,1), i.e. $m^i{}_j$

❑ Tensors can be categorized depending on certain transformation rules:
- Covariant Tensor: indices are low
- Contravariant Tensor: indices are high
- Tensor can be mixed rank (made from covariant and contravariant indices)
- Euclidean 3D space: Covariant = Contravariant = Cartesian Tensors
- 注意 $m^i{}_{jk} \neq m^{ij}{}_k$

4.2: Geometry, Metrics & Tensors

- **A little bit about Tensors - 2**

❑ Covariant Tensor Transformation:

Consider Scalar quantity ϕ. Scalar ➡ invariant under coordinate transformations $\phi(x_i) = \phi(x'_i)$

$$e.g. \quad \phi(x_1, x_2, x_3) = \phi(x'_1, x'_2, x'_3)$$

Taking the gradient $\nabla\phi = \frac{\partial\phi}{\partial x_1}\hat{\mathbf{x}}_1 + \frac{\partial\phi}{\partial x_2}\hat{\mathbf{x}}_2 + \frac{\partial\phi}{\partial x_3}\hat{\mathbf{x}}_3$ this normal has components $A_i = \frac{\partial\phi}{\partial x_i}$

Which should not change under coordinate transformation, so $A'_i = \frac{\partial\phi}{\partial x'_i} = \frac{\partial\phi}{\partial x'_i}$

Also have the relation $\frac{\partial\phi}{\partial x'_i} = \frac{\partial\phi}{\partial x_j}\frac{\partial x_j}{\partial x'_i}$

So for, $A_i = \frac{\partial\phi}{\partial x_i}$ $A'_i = \frac{\partial'\phi}{\partial x'_i}$ $A_j = \frac{\partial\phi}{\partial x_j}$ etc……

$$A'_i = \frac{\partial'\phi}{\partial x'_i} = \frac{\partial x_j}{\partial x'_i}\frac{\partial\phi}{\partial x_j} = \frac{\partial x_j}{\partial x'_i}A_j \qquad A'_i = a_{i,j}A_j \qquad a_{i,j} = \frac{\partial x_j}{\partial x'_i}$$

Generalize for 2nd rank Tensors,

Covariant 2nd rank tensors are animals that transform as :-

$$\boxed{A'_{ij} = \frac{\partial x_k}{\partial x'_i}\frac{\partial x_l}{\partial x'_j}A_{kl}}$$

4.2: Geometry, Metrics & Tensors

- **A little bit about Tensors - 3**

❑ Contravariant Tensor Transformation:

Consider a curve in space, parameterized by some value, r

Coordinates of any point along the curve will be given by: $x^i(r)$

Tangent to the curve at any point is given by: $A^i = \frac{dx^i}{dr}$

The tangent to a curve should not change under coordinate transformation, so $A'^i = \frac{dx'^i}{dr}$

Also have the relation $\frac{dx'^i}{dr} = \frac{\partial x'^i}{\partial x_j}\frac{dx^j}{dr}$

So for, $A^i = \frac{dx^i}{dr}$ $A'^i = \frac{dx'^i}{dr}$ $A^j = \frac{dx^j}{dr}$ etc……

$$A'^i = \frac{dx'^i}{dr} = \frac{\partial x'^i}{\partial x_j}\frac{dx^j}{dr} = \frac{\partial x'^i}{\partial x_j}A^j \qquad A'^i = a^{i,j}A^j \qquad a^{i,j} = \frac{\partial x'^i}{\partial x^j}$$

Generalize for 2nd rank Tensors,

Contravariant 2nd rank tensors are animals that transform as :-

$$\boxed{A'^{ij} = \frac{\partial x'_i}{\partial x_k}\frac{\partial x'_j}{\partial x_l}A^{kl}}$$

4.2: Geometry, Metrics & Tensors

- **More about Tensors - 4**

❑ Index Gymnastics

❑ Raising and Lowering indices:

$$g_{ij}A^i = A_j$$

Transform covariant to contravariant form by multiplication by metric tensor, $g_{i,j}$

$$g^{ij}A_i = A^j$$

❑ Einstein Summation:

$$\sum_i a_i a_i = a_i a_i$$

Repeated indices (in sub and superscript) are implicitly summed over

$$\sum_k A_{ik}B^k = A_{ik}B^k$$

$$\sum_{i,k} A_{ik}B^iB^k = A_{ik}B^iB^k$$

Einstein c.1916: "I have made a great discovery in mathematics; I have suppressed the summation sign every time that the summation must be made over an index which occurs twice..."

Thus, the metric $ds^2 = \sum_{i,j=0}^{n} g_{ij}dx^i dx^j$ May be written as $\boxed{ds^2 = g_{ij}dx^i dx^j}$

4.2: Geometry, Metrics & Tensors

- **More about Tensors - 5**

❑ Tensor Manipulation

❑ Tensor Contraction:

Set unlike indices equal and sum according to the Einstein summation convention.
Contraction reduces the tensor rank by 2.
For a second-rank tensor this is equivalent to the Scalar Product

$$A^i B_j$$

$$i = j$$

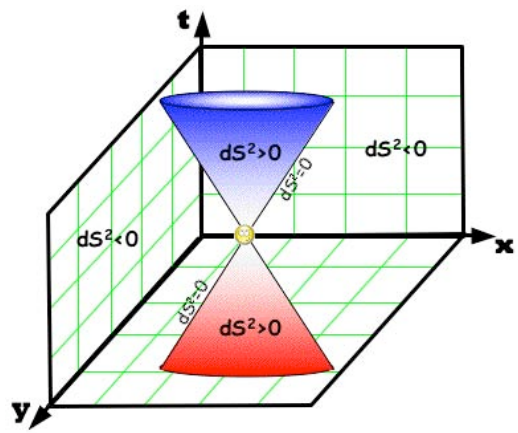
$$A^i B_j = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3$$

Minkowski Spacetime (Special Relativity)

$$g_{ij} A^i A^j < 0 \quad \text{spacelike}$$

$$g_{ij} A^i A^j > 0 \quad \text{timelike}$$

$$g_{ij} A^i A^j = 0 \quad \text{null}$$



4.2: Geometry, Metrics & Tensors

- **More about Tensors - 6**

❑ Tensor Calculus

❑ The Covariant Derivative of a Tensor:

Derivatives of a Scalar transform as a vector

$$A_i = \frac{\partial \phi}{\partial x_i}$$

How about derivative of a Tensor ?? Does $\frac{\partial A_i}{\partial x^j}$ transform as a Tensor?

Lets see.....
Take our definition of Covariant Tensor $A'_i = \frac{\partial x^j}{\partial x'^i} A_j$

Problem

$$\frac{\partial A'_i}{\partial x'^m} = \frac{\partial x^j}{\partial x'^i}\frac{\partial x^n}{\partial x'^m}\frac{\partial A_j}{\partial x^n} + \frac{\partial^2 x^j}{\partial x'^m \partial x'^i} A_j$$

Implies transformation coefficients vary with position

Correction Factor: Γ^k_{ij} such that for $\frac{\partial A_i}{\partial x^j}$, $\delta A_i = \Gamma^k_{ij} A_k \delta x^j$

- Christoffel Symbols
- Defines parallel vectors at neighbouring points
- Parallelism Property: affine connection of S-T
- Christoffel Symbols = fn(spacetime)

Redefine the covariant derivative of a tensor as;

$$\boxed{A_{i;j} \equiv \frac{\partial A_i}{\partial x^j} - \Gamma^k_{ij} A_k \equiv A_{i,j} - \Gamma^k_{ij} A_k}$$

, normal derivative
; covariant derivative

4.2: Geometry, Metrics & Tensors

- **Riemann Geometry**

higher rank ➟ extra Christoffel Symbol

Covariant Differentiation of Metric Tensor $g_{ik;l} \equiv g_{ik,l} - \Gamma^{p}_{il} g_{pk} - \Gamma^{p}_{kl} g_{ip}$ 1

General Relativity formulated in the non-Euclidean Riemann Geometry

➟ Simplifications $\Gamma^{i}_{kl} = \Gamma^{i}_{lk}$ $\qquad g_{ik;l} = 0$

1 ➟ 40 linear equations - ONE Unique Solution for Christoffel symbol $\Gamma^{i}_{kl} = \frac{g^{im}}{2}\left(g_{mk,l} + g_{ml,k} - g_{kl,m}\right)$

SPECIAL RELATIVITY $ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$ $\qquad \int_{path} ds = constant$ Shortest distance between 2 points is a straight line.

GENERAL RELATIVITY $ds^2 = g_{ij} dx^i dx^j$ $\qquad \delta \int_{path} ds = 0$ But lines are not straight (because of the metric tensor)

Gravity is a property of Spacetime, which may be curved

Path of a free particle is a geodesic where $\frac{d^2 x^i}{ds^2} + \Gamma^{i}_{kl} \frac{dx^k}{ds} \frac{dx^l}{ds} = 0$

4.2: Geometry, Metrics & Tensors

- **The Riemann Tensor**

Recall

correction factor for the transportation of components of parallel vectors between neighbouring points $\Delta A_i = \Gamma^l_{ik} A_l \Delta x^k$

Such that we needed to define the Covariant derivative $A_{i;k} \equiv \frac{\partial A_i}{\partial x^k} - \Gamma^l_{ik} A_l \equiv A_{i,k} - \Gamma^l_{ik} A_l$

To transport a vector without the result being dependent on the path,

Require a vector $A_i(x^k)$ such that - $\frac{\partial A_i}{\partial x^k} = \Gamma^l_{ik} A_l$ 1

$\frac{\partial}{\partial x^n}$ 1 $\quad \frac{\partial^2 A_i}{\partial x^n \partial x^k} = \frac{\partial}{\partial x^n}\left(\Gamma^l_{ik} A_l\right) = \frac{\partial \Gamma^l_{ik}}{\partial x^n} A_l + \Gamma^l_{ik} \frac{\partial A_l}{\partial x^n} = \left[\frac{\partial \Gamma^m_{ik}}{\partial x^n} + \Gamma^l_{ik} \Gamma^m_{ln}\right] A_m$

Interchange differentiation w.r.t x_n & x_k and use $A_{i,nk} = A_{i,kn}$

Necessary condition for 1 $\quad 0 = \frac{\partial \Gamma^m_{ik}}{\partial x^n} - \frac{\partial \Gamma^m_{in}}{\partial x^k} + \Gamma^l_{ik} \Gamma^m_{ln} - \Gamma^l_{in} \Gamma^m_{lk} \equiv R_i^{\ m}{}_{kn}$

Can show: $A_{i;nk} - A_{i;kn} \equiv R_i^{\ m}{}_{kn} A_m$ Where RHS=LHS=Tensor $\quad R_i^{\ m}{}_{kn}$ is the Riemann Tensor

- Defines geometry of spacetime (=0 for flat spacetime).
- Has 256 components but reduces to 20 due to symmetries.

4.2: Geometry, Metrics & Tensors

- **The Einstein Tensor**

- Lowering the second index of the Riemann Tensor, define $$g_{ij}R^{j}_{klm} = R_{iklm}$$

- Contracting the Riemann Tensor gives the Ricci Tensor describing the curvature $$R_{kl} = g^{im}R_{iklm} \equiv R^{m}{}_{klm}$$

- Contracting the Ricci Tensor gives the Scalar Curvature (Ricci Scalar) $$R = g^{kl}R_{kl}$$

$R_{i}{}^{m}{}_{kn}$ is the Riemann Tensor

- Define the Einstein Tensor as $$G_{ik} = R_{kl} - \frac{1}{2}g_{ik}R$$

- Symmetry properties of R_{iklm} $$R_{iklm;n} + R_{iknl;m} + R_{ikmn;l} = 0$$ Bianchi Identities

Multyplying by $g^{im}g^{kn}$ and using above relations ➠ $$\left[R^{ik} - \frac{1}{2}g_{ik}R\right]_{;k} \equiv 0$$

or…… The Einstein Tensor has ZERO divergence

4.3: Einstein's Theory of Gravity

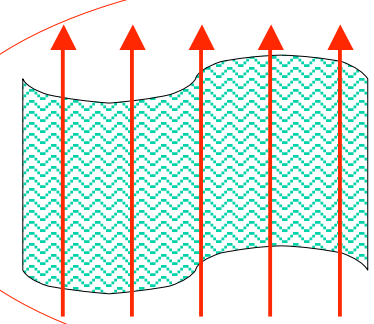
- **The Energy Momentum Tensor**

Define Energy Momentum Tensor

$$T^{ik} \begin{cases} T^{00} & \text{Density of mass /unit vol.} \\ T^{i0} & \text{Density of } j^{th} \text{ component of momentum/unit vol.} \\ T^{ik} & \text{k-component of flux of j component of momentum} \end{cases}$$

T^{ik} The flux of the i momentum component across a surface of constant x^k

- Conservation of Energy and momentum - $T^k{}_{i;k}=0$



Dust approximation
World lines almost parallel
Speeds non-relativistic

❑ Non relativistic particles (Dust)

$\rho_o = \rho_o(x)$ - Proper density of the flow

$u^i = dx^i/dt$ - 4 velocity of the flow

- In comoving rest frame of N dust particles of number density, n

$$T^{ik} = T^{00} = \sum_{N}^{Vol} m_N c^2 = nmc^2 = \rho_o c^2$$

Consider Units

Density, ρ = mass/vol Momentum, p = energy/vol

In general, $T = p \quad u = mnu \quad u = \rho u \quad u$

- In any general Lorentz frame $T^{ik} = \rho u^i u^k$

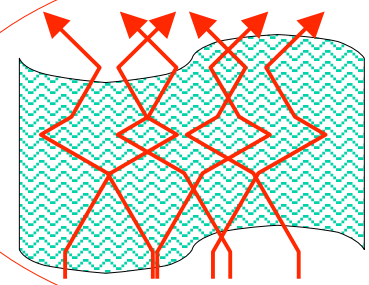
Looks like classical Kinetic energy

4.3: Einstein's Theory of Gravity

- **The Energy Momentum Tensor**

❑ Relativistic particles (inc. photons, neutrinos)

Particle 4 Momentum, $p^i = \left(\frac{E}{c}, \vec{p}\right)$ $\qquad E^2 = c^2\vec{p}^2 + m^2c^4 \Rightarrow \vec{p}^2c^2$



Relativistic approximation
Particles frequently colliding
Pressure
Speeds relativistic

$$T^{00} = \sum_N^{Vol} E_N = \rho \qquad \rho \text{- energy density}$$

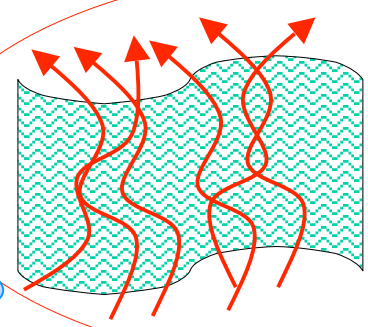
$$T^{11} = T^{22} = T^{22} = \sum_N^{Vol} \frac{\vec{p}^2c^2}{3E} \Rightarrow \frac{1}{3}\rho$$

1/3 since random directions

❑ Perfect Fluid

General Particle 4 Momentum, $\vec{p}^0 = \gamma mc^2$

$\vec{p}^{\alpha} = \gamma m\vec{v} \quad (\alpha = 1,2,3)$



Fluid approximation
Particles colliding - Pressure
Speeds nonrelativistic

In rest frame of particle

$$T^{00} = \sum_N^{Vol} \gamma mc^2 \Rightarrow \rho c^2 \qquad T^{11} = T^{22} = T^{22} = \sum_N^{Vol} \frac{\gamma\vec{p}^2c^2}{3} \Rightarrow P$$

Looks like gas law

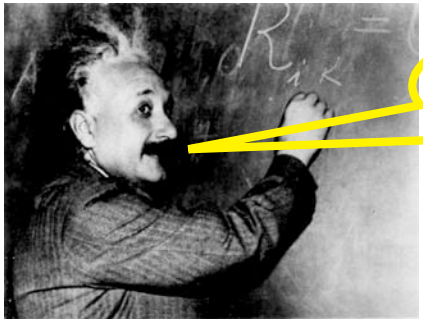
For any reference frame with fluid 4 velocity = u

$$T^{ik} = (P + \rho c^2)u^i u^k \Rightarrow g^{ik}P$$

Pressure is important

4.3: Einstein's Theory of Gravity

- **The Einstein Equation**



After trial and error !!
Einstein proposed his famous equation

Energy Momentum Tensor (matter in the Universe)

$$T^{ik} = (P + \rho c^2)u^i u^k - g^{ik} P$$

Einstein Tensor (Geometry of the Universe)

$$G_{ik} = R_{kl} - \frac{1}{2} g_{ik} R$$

ご結婚

$$G_{ik} = \kappa T^{ik} = R_{kl} - \frac{1}{2} g_{ik} R$$

- Classical limit
- must reduce to Poisson's eqn. for gravitational potential $\nabla^2 \phi = -4\pi G \rho$

$$\kappa = \frac{8\pi G}{c^4}$$

Basically: Relativistic Poisson Equation

$$G_{ik} = R_{kl} - \frac{1}{2} g_{ik} R = \frac{8\pi G}{c^4} T^{ik}$$

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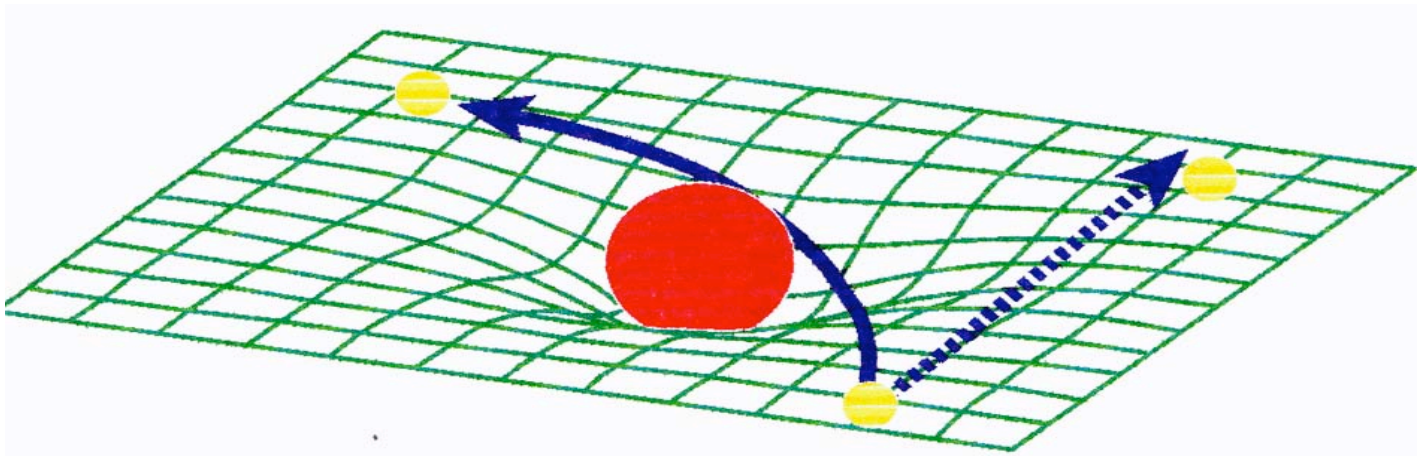
4.3: Einstein's Theory of Gravity

- **The Einstein Equation**

"Matter tells space how to curve. Space tells matter how to move."



Fabric of the Spacetime continuum and the energy of the matter within it are interwoven

注意

$$G_{ik} = R_{kl} - \frac{1}{2} g_{ik} R = \frac{8\pi G}{c^4} T^{ik}$$

Does not give a static solution !!

Einstein inserted the Cosmological Constant term Λ

$$G_{ik} = R_{kl} - \frac{1}{2} g_{ik} R - \Lambda g_{ik} = \frac{8\pi G}{c^4} T^{ik}$$

"The Biggest Mistake of My Life "

4.3: Einstein's Theory of Gravity

- **Solutions of the metric** $ds^2 = \sum_{i,j=0}^{3} g_{i,j} dx^i dx^j$ just spatial coords $ds^2 = \sum_{\alpha,\beta=1}^{3} g_{\alpha,\beta} dx^\alpha dx^\beta$

- Cosmological Principle:
  - Isotropy ➡ $g_{\alpha,\beta} = \delta_{\alpha,\beta}$, only take $\alpha=\beta$ terms
  - Homogeneity ➡ $g_{0,\alpha} = g_{0,\alpha} = 0$

The Metric: $-ds^2 = g_{00}(dx^0)^2 + g_{\alpha\beta} dx^\alpha dx^\beta = g_{00}(dx^0)^2 + dl^2$

Proper time interval: $dt = \sqrt{-g_{00}}\,\frac{dx^0}{c}$ $\quad ds^2 = c^2 dt^2 - dl^2$

How about $g_{\alpha,\beta} dx^\alpha dx^\beta$ *?*

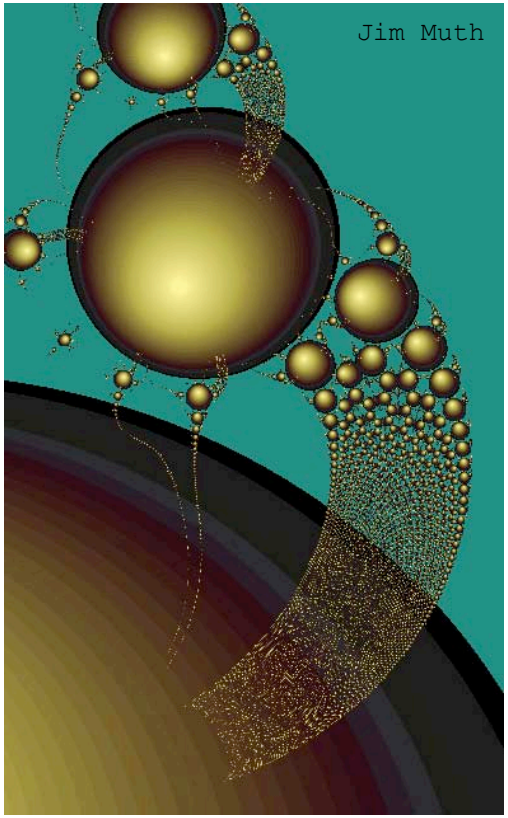
embed our curved, isotropic 3D space in hypothetical 4-space.

➠ our 3-space is a hypersurface defined by $(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = a^2$

➠ length element becomes: $dl^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2$

Constrain dl to lie in 3-Space
(eliminate x^4)

$$dl^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + \frac{x^1 dx^1 + x^2 dx^2 + x^3 dx^3}{a^2 - (x^1)^2 - (x^2)^2 - (x^3)^2}$$



4.3: Einstein's Theory of Gravity

- **Solutions of the metric**

$$x^1 = r\sin\theta\cos\phi$$
$$x^2 = r\sin\theta\sin\phi$$
$$x^3 = r\cos\theta$$

Introduce spherical polar identities in 3 Space

$$dl^2 = dr^2 + r^2\sin^2\theta d\phi^2 + r^2 d\theta^2 + \frac{r^2 dr^2}{a^2 - r^2} = \frac{dr^2}{1 - r^2/a^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

I. Multiply spatial part by arbitrary function of time $R(t)$: wont affect isotropy and homogeneity because only a f(t)
II. Absorb elemental length into $R(t)$ ➡r becomes dimensionless
III. Re-write $a^{-2} = k$

dS

$$dS^2 = c^2dt^2 - R^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

The Robertson-Walker Metric

- r, θ, ϕ are co-moving coordinates and don't changed with time - They are SCALED by $R(t)$
- *t is the cosmological proper time or cosmic time - measured by observer who sees universe expanding around him*
- The co-ordinate, r, can be set such that $k = -1, 0, +1$

4.3: Einstein's Theory of Gravity

- **The Robertson-Walker Metric and the Geometry of the Universe**

$$dS^2 = c^2 dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

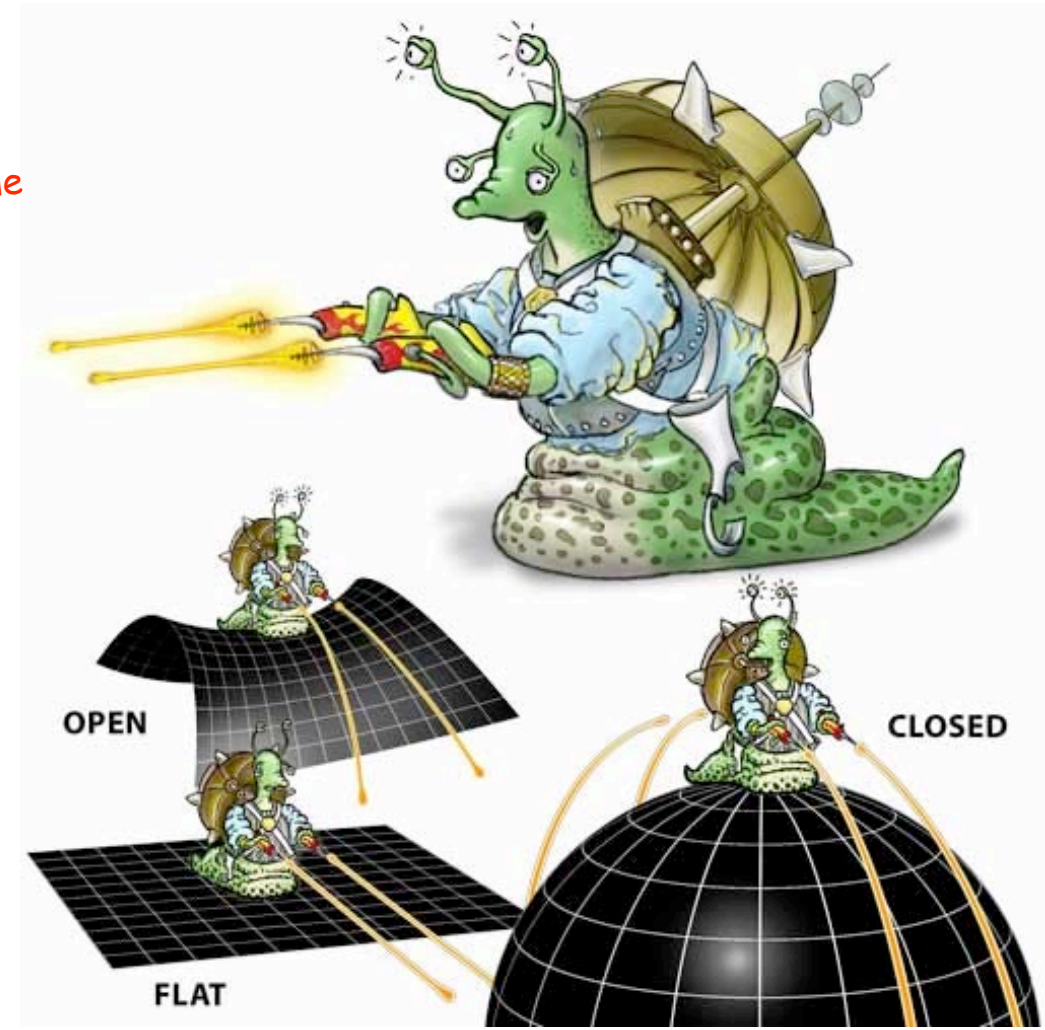
The Robertson-Walker Metric

k ~ defines the curvature of space time

k = 0 : Flat Space

k = -1 : Hyperbolic Space

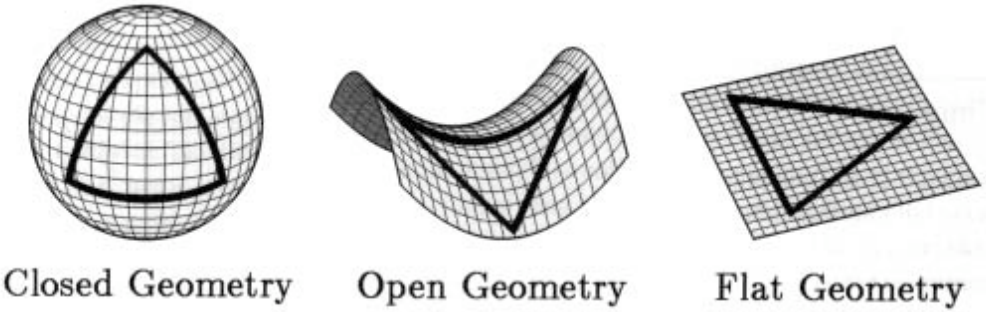
k = +1 : Spherical Space

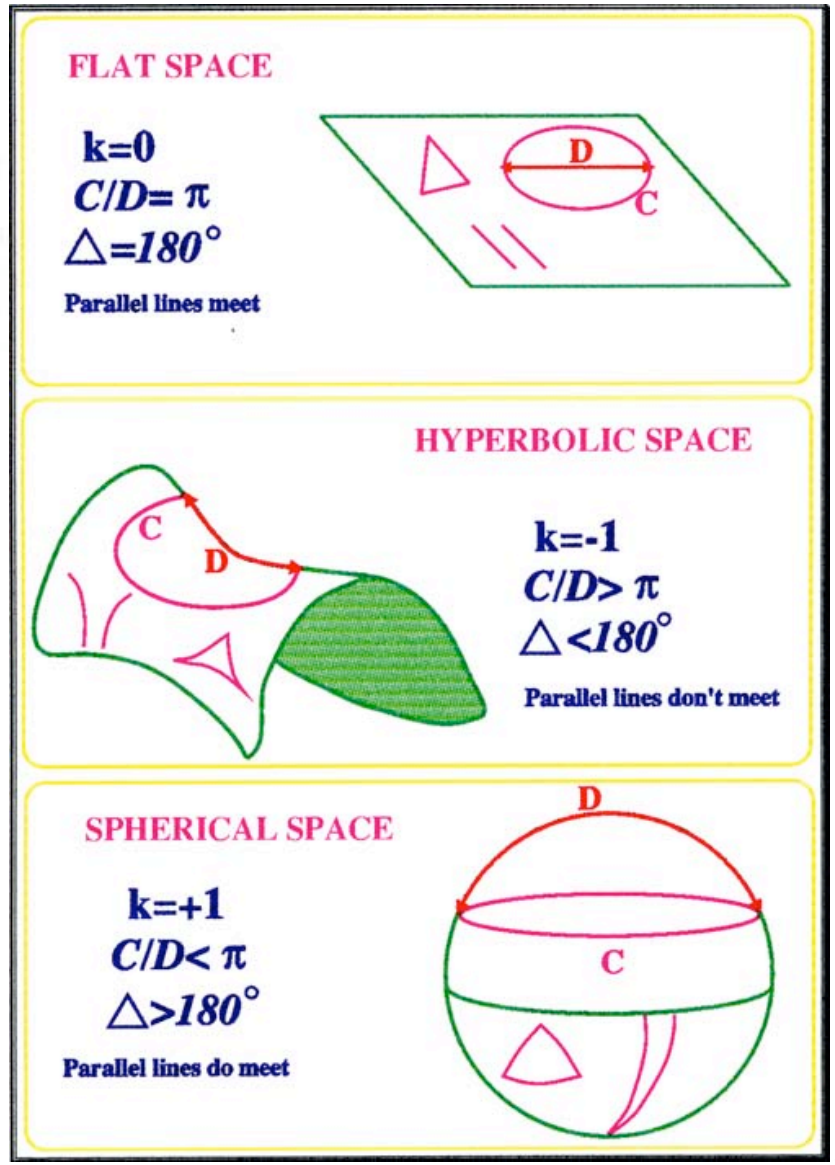


http://map.gsfc.nasa.gov/

4.3: Einstein's Theory of Gravity

- **The Robertson-Walker Metric and the Geometry of the Universe**





$k = 0$: *Flat Space*

$k = -1$: *Hyperbolic Space*

$k = +1$: *Spherical Space*

4.3: Einstein's Theory of Gravity

- **The Friedmann Equations in General Relativistic Cosmology**

Solve Einstein's equation :

$$G_{ik} = R_{kl} - \frac{1}{2} g_{ik} R = \frac{8\pi G}{c^4} T^{ik}$$

Need:
- metric $\mathbf{g_{ik}}$
- energy tensor $\mathbf{T^{ik}}$

Metric is given by R-W metric:

$$dS^2 = c^2 dt^2 - S^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

Use S so don't confuse with Tensors

Co-ordinates $\quad x^0 = ct \quad x^1 = r \quad x^2 = \theta \quad x^3 = \phi$

Non-zero g^{ik}

$$g_{00} = 1 \quad g_{11} = -\frac{S^2}{1-kr^2} \quad g_{22} = -S^2 r^2 \quad g_{33} = -S^2 r^2 \sin^2\theta$$

$$g_{ik} = \frac{1}{g^{ik}} \qquad g_{00} = 1 \quad \sqrt{-g} = -\frac{Sr\sin\theta}{\sqrt{1-kr^2}}$$

$$\Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} = \frac{\dot{S}}{cS}$$

Non-zero Γ^i_{kl}

$$\Gamma^0_{11} = \frac{S\dot{S}}{c(1-kr^2)} \quad \Gamma^0_{22} = \frac{S\dot{S}r^2}{c} \quad \Gamma^0_{33} = \frac{S\dot{S}r^2\sin^2\theta}{c}$$

$$\Gamma^1_{11} = \frac{kr}{1-kr^2} \quad \Gamma^1_{22} = -r(1-kr^2) \quad \Gamma^1_{33} = -r(1-kr^2)\sin^2\theta$$

$$\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r} \quad \Gamma^2_{33} = -\sin\theta\cos\theta \quad \Gamma^3_{23} = \cot\theta$$

4.3: Einstein's Theory of Gravity

- **The Friedmann Equations in General Relativistic Cosmology**

Non-zero components of Ricci Tensor R_{ik}

$$R_0^0 = \frac{3}{c^2}\frac{\ddot{S}}{S} \qquad R_1^1 = R_2^2 = R_3^3 = \frac{1}{c^2}\left(\frac{\ddot{S}}{S} + \frac{2\dot{S}^2 + 2kc^2}{S^2}\right)$$

The Ricci Scalar R

$$R = \frac{6}{c^2}\left(\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2}\right)$$

$$G_0^0 = R_0^0 - \frac{R}{2} = -\frac{3}{c^2}\left(\frac{\dot{S}^2 + kc^2}{S^2}\right)$$

Non-zero components of Einstein Tensor G_i^k

$$G_1^1 = G_2^2 = G_3^3 \equiv R_1^1 - \frac{R}{2} = -\frac{1}{c^2}\left(2\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2}\right)$$

Sub ➟ Einstein

$$\frac{\dot{S}^2 + kc^2}{S^2} = \frac{8\pi G}{3c^2}T_0^0$$

$$2\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2} = \frac{8\pi G}{3c^2}T_1^1 = \frac{8\pi G}{3c^2}T_2^2 = \frac{8\pi G}{3c^2}T_3^3$$

4.3: Einstein's Theory of Gravity

- **The Friedmann Equations in General Relativistic Cosmology**

Non-zero components of Energy Tensor T^i_k

1 $$2\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2} = \frac{8\pi G}{3c^2}T^1_1 = \frac{8\pi G}{3c^2}T^2_2 = \frac{8\pi G}{3c^2}T^3_3$$

2 $$\frac{\dot{S}^2 + kc^2}{S^2} = \frac{8\pi G}{3c^2}T^0_0$$

$T^0_0 = \rho$

$T^1_1 = T^2_2 = T^3_3 = -P$

$$\frac{\partial}{dt}\left(\frac{d(\rho S^3)}{dS} + 3PS^2 = 0\right)$$ 2 3

actually implied by $T^k_{i;k}=0$

Assume Dust:
- $P = 0$
- $\rho = \rho c^2$

$$\frac{d(\rho S^3)}{dS} = 0 \quad \Rightarrow \quad \rho = \rho_o\left(\frac{S_o}{S}\right)^3 \quad \Rightarrow \quad T^0_0 = \rho_o c^2\left(\frac{S_o}{S}\right)^3, \qquad T^1_1 = 0$$

3 → 1 $$2\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2} = 0$$

3 → 2 $$\frac{\dot{S}^2 + kc^2}{S^2} = \frac{8\pi G\rho_o}{3c^2}\left(\frac{S_o}{S}\right)^3$$

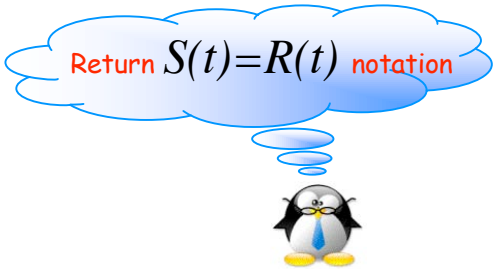
4.3: Einstein's Theory of Gravity

- **The Friedmann Equations in General Relativistic Cosmology**

1 $$2\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2} = 0$$

2 $$\frac{\dot{S}^2 + kc^2}{S^2} = \frac{8\pi G\rho_o}{3c^2}\left(\frac{S_o}{S}\right)^3$$

3 $$\rho = \rho_o\left(\frac{R_o}{R}\right)^3$$



Finally .....

2 ➡ $$\dot{R}^2 = \frac{8\pi G\rho}{3}R^2 - kc^2 \qquad \left(\rho + \frac{\Lambda R^2}{3}\right)$$

⬇

3 ➡ $$\ddot{R} = -\frac{4\pi G\rho}{3}R \qquad \left(\rho + \frac{\Lambda R}{3}\right)$$

Our old friends the Friedmann Equations

4.4: Summary

- **The Friedmann Equations in General Relativistic Cosmology**
  - *We have come along way today!!!*

$$G_{ik} = R_{kl} - \frac{1}{2} g_{ik} R = \frac{8\pi G}{c^4} T^{ik}$$

Deriving the necessary components of The Einstein Field Equation
- Spacetime and the Energy within it are symbiotic
- The Einstein equation describes this relationship

$$dS^2 = c^2 dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

The Robertson-Walker Metric defines the geometry of the Universe

$$\ddot{R} = -\frac{4\pi G\rho}{3} R \left[\rho + \frac{3p}{c^2}\right]$$

$$\dot{R}^2 = \frac{8\pi G\rho}{3} R^2 - kc^2 \left[\rho + \frac{3p}{c^2}\right]$$

The Friedmann Equations describe the evolution of the Universe

THESE WILL BE OUR TOOLBOX FOR OUR COSMOLOGICAL STUDIES

4.4: Summary

Fundamental Cosmology
4. General Relativistic Cosmology

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Fundamental Cosmology
5. The Equation of state
& the Evolution of the Universe