Inertia moment for a Cone

0.1 Prerequisites:

0.1.1 Volume of a cone

$$V = \frac{1}{3}\pi r 2h \tag{0.1}$$

Because density $\rho = \frac{M}{V}$ Comes out that

$$\rho = \frac{3M}{\pi r^2 h} \tag{0.2}$$

0.1.2 Moment of inertia of a solid body

$$I = mr^2 \tag{0.3}$$

0.1.3 Moment of inertia of a disk

$$I = \frac{1}{2}mr^2 \tag{0.4}$$

0.1.4 Torque

$$\tau = I\alpha \tag{0.5}$$

Bold letters indicate vector quantities.

$$\tau = \mathbf{r} \times \mathbf{F} = m[\mathbf{r} \times (\boldsymbol{\alpha} \times \mathbf{r})] = mr^2\boldsymbol{\alpha} = I\boldsymbol{\alpha} = I\alpha\mathbf{e} \tag{0.6}$$

Since

$$\frac{d\mathbf{v}}{dt} = \mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} \tag{0.7}$$

where I is the moment of inertia and α is the angle of rotation
The SI units of I are N.m

0.1.5 Volume of a sphere

$$V = \int_{0}^{r} A(r)dr \tag{0.8}$$

where $A(r) = 4\pi r^2$ and that is

$$V = \int_{0}^{r} 4\pi r^2 = \frac{4\pi r^3}{3} \tag{0.9}$$

0.1.6 Angular momentum L

I = moment of inertia and ω is the angle of rotation.

$$L = I\omega \tag{0.10}$$

For a cylindrical section of the cone we have:

$$dI = \frac{1}{2}dmr^2 \tag{0.11}$$

where dI is the inertia for a infinitely thin slice, dm the infinitely small mass and r the radius of the cylinder

Substituting dm by ρdv where ρ is the density and dv the infinitely small volume, we get

$$dI = \frac{1}{2}\rho dvr^2 \tag{0.12}$$

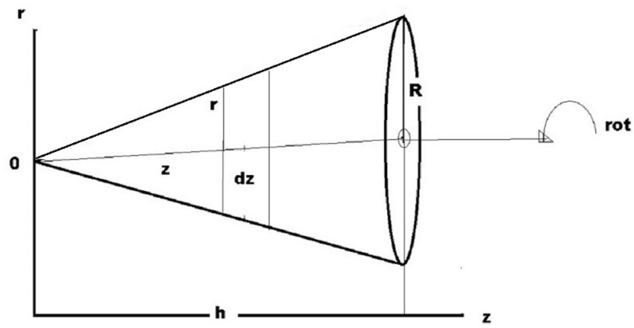

Figure 0.1: rotating cone about z

But $\rho dv = \pi r^2 dz$ Then

$$dI = \frac{1}{2}\rho(\pi r^2 dz) r^2 \tag{0.13}$$

Now substituting r with the equivalent from figure 1 quantity $R\frac{Z}{h}$

we get

$$dI = \frac{1}{2}\rho \left(\pi \left(R\frac{Z}{h}\right)^4\right) dz \tag{0.14}$$

and substituting ρ with equivalent from cone volume

$$dI = \frac{1}{2}\left(\frac{3M}{\pi R^2 h}\right)\pi\left(R\frac{Z}{h}\right)^4 dz = \frac{3}{2}MR^2\frac{Z^4}{h^5}dz \tag{0.15}$$

And integrating we get

$$dI = \int \frac{3}{2} M R^2 \frac{Z^4}{h^5} dz \tag{0.16}$$

$$= \left(\frac{3}{2} \frac{M R^2}{h^5} \right) \int z^4 dz \tag{0.17}$$

$$= \left[\left(\frac{3}{2} \frac{M R^2}{h^5} \right) \left(\frac{1}{5} h^5 \right) \right] \tag{0.18}$$

$$= \frac{3}{10} M R^2 \tag{0.19}$$